Speed and Shape of Solitary Waves in Two-electron Plasmas with Relativistic Warm Ions

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Large amplitude solitary waves are investigated in a relativistic plasma with finite ion-temperature and two temperature isothermal electrons. Sagdeev's pseudopotential is determined in terms of the ion speed u. It is found that there exists a critical value of u_0, the value of u at which $(u')^2 = 0$, beyond which the solitary waves cease to exists. The critical value also depends on parameters like the soliton velocity v, the fraction of the cold electron concentration μ, or the ratio of the cold and hot electron temperatures β.

Key words: Solitary Wave; Pseudopotential; Warm Ions.

1. Introduction

Ion-acoustic solitary waves have been studied theoretically and experimentally by several authors [1 – 20] during the last three decades. Washimi and Taniuti [1] were the first to study the propagation of ion acoustic solitary waves in a cold plasma. But Ikezi, Tailor and Baker [20] first experimentally discovered ion-acoustic solitons (IAS) and double layers in plasmas. The history of studies on ion acoustic waves is quite long, and many authors worked in this field. To keep a close relation between the theory and experiment, many authors introduced various parameters *e.g* negative ion concentration [3, 4], ion temperature [8 – 10], non-isothermality [14], etc. in their plasma models. Most of them derived Kortweg de Vries (KdV) or Modified KdV (MKdV) type equations which explained the characteristics like width, amplitude or velocity of the solitary wave nicely. Tran and Hirt [11] investigated ion waves in multi component plasmas. But in their model they ignored the possibility of negative ion interactions with ion waves, and so the model was to some extent limited. Das and Tagare [3] and Das [4] studied the effects of negative ions on solitons and showed that there exists a critical density of negative ions at which the solitary waves becomes infinitely large. Many authors [8 – 10] have also extended their investigation to study the effect of finite ion-temperature on the amplitude and width of solitary waves. Solitary waves and double layers in two-temperature electron plasmas have also been investigated by Das, Pal and Karmakar [6] and Roychoudhury, Bhattacharyya and Varshini [7]. The evolution of solitary waves was also studied in the frame work of the KP (Kodomtsev-Petviashvilli) equation by Das and Sen [5]. To derive KdV, MKdV or KP type equations most of the authors used the reductive perbative technique (RPT). A few years ago Malfliet and Wieers [21] reviewed the studies on solitary waves in plasmas and found that the RPT, which is based on the assumption of smallness of amplitude can explain only small amplitude solitary waves. But large amplitude solitary waves also exist in nature. Nakamura et al. [18] observed large amplitude solitary waves in laboratory plasmas. So to study large amplitude solitary waves one has to employ a nonperturbative approach. Sagdeev's [22] pseudopotential method is one such method to obtain exact solitary wave solution which has been successfully applied in various cases [23 – 25] including multicomponent and multi-dimentional plasmas.

More recently Johnston and Epstein [26] studied the nonlinear ion-acoustic solitary waves in a cold collisionless plasma by the direct analysis of the field equations. They observed that a very small change in the initial condition destroys the oscillatory behaviour of the solitary waves. Recently Maitra and Roychoudhury [29] studied dust-acoustic solitary waves using this technique.

In this paper our aim is to study large amplitude solitary waves in a plasma with one relativistic warm ion and two isothermal electrons. The motivation for this study was the following:

1) Relativistic effects play a significant role in the study of solitary waves when the speeds of the particles are comparable to the velocity of light. For example, ions with very high speed are frequently observed in the solar atmosphere and interplanetary space. High energy ion beams are observed in the plasma sheet boundary layer of the earth's atmosphere and in the Van Allen radiation belts. See [30, 31].

2) Consideration of two electrons in plasmas leads to very interesting results in the propagation of solitary waves. Two-temperature electron plasmas can be produced in the laboratory. Since two-temperature electron plasmas observed both in the laboratory and in space, it is important to study solitary waves in such plasma. See [32, 33].

Here we will study how μ, the concentration of hot and cold electrons plays a role on the region of soliton solutions. We will also study the effect of β, the ratio of the cold and hot electron temperatures and v, the soliton velocity, on the existence and shape of solitary waves.

The organization of the paper is as follows. In Sect. 2 the basic equations are written, considering two iso thermal electrons and a relativistic warm ion. The governing second order ordinary differential equation is derived. Section 3 is kept for results and discussions, and Sect. 4 for conclusion.

2. Basic Equations

Our analysis is based on the continuity and momentum fluid equation for ions and electrons, and Poisson's equation, as are given below (see [6, 7]):

$$\frac{\partial n}{\partial t}+\frac{\partial (nu)}{\partial x}=0, \tag{1}$$

$$\frac{\partial u\gamma}{\partial t}+u\frac{\partial u\gamma}{\partial x}+\frac{\sigma}{n}\frac{\partial p}{\partial x}=-\frac{\partial \phi}{\partial x}, \tag{2}$$

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+3p\frac{\partial u\gamma}{\partial x}=0, \tag{3}$$

$$\frac{\partial^2 \phi}{\partial x^2}=n_{\mathrm{ec}}+n_{\mathrm{eh}}-n, \tag{4}$$

where

$$n_{\mathrm{ec}}=\mu e^{\phi/(\mu+\nu\beta)}, \tag{5}$$

$$n_{\mathrm{eh}}=\nu e^{\beta\phi/(\mu+\nu\beta)}, \tag{6}$$

$$\gamma=\frac{1}{\sqrt{1-u^2/c^2}}. \tag{7}$$

μ is the fraction of cold electrons at the temperature T_{ec} and ν is the fraction of hot electrons at T_{h}. Also $\mu+\nu=1$, and $\beta=T_{\mathrm{c}}/T_{\mathrm{h}}$ where T_{c} and T_{h} are temperature of the cold and hot electrons, respectively. n, n_{ec} and n_{eh} are the normalized ion, cold electron and hot electron density, respectively. u, the flow velocity of the ions, and c, the velocity of light are normalized to $(\kappa T_{\mathrm{eff}}/m_{\mathrm{i}})^{1/2}$. p denotes the ion pressure normalized to $(n_0\kappa T_{\mathrm{i}})^{-1}$, ϕ is the electrostatic potential normalized to $\kappa T_{\mathrm{eff}}/e$. Space and time are normalized by the Debye length $\lambda_{\mathrm{D}}=(\kappa T_{\mathrm{eff}}/4\pi e^2 n_0)^{1/2}$ and ion plasma frequency $\omega_{\mathrm{i}}^{-1}=(m_{\mathrm{i}}/4\pi e^2 n_0)^{1/2}$, respectively. $\sigma=T_{\mathrm{i}}/T_{\mathrm{eff}}$, where T_{i} is the ion-temperature, and $T_{\mathrm{eff}}=(T_{\mathrm{c}}+T_{\mathrm{h}})/(\mu T_{\mathrm{h}}+\nu T_{\mathrm{c}})$. e is the electric charge.

In order to investigate the properties of solitary wave solutions of equations (1) to (7), we assume that all dependent variables depend on a single independent variable $\xi=x-vt$, where v is the velocity of the solitary wave and the variable ξ is the special coordinate in the coordinate system moving with the solitary wave velocity.

Now equations (1) – (4) reduce to

$$-v\frac{\mathrm{d}n}{\mathrm{d}\xi}+\frac{\mathrm{d}(nu)}{\mathrm{d}\xi}=0, \tag{8}$$

$$-v\frac{\mathrm{d}u\gamma}{\mathrm{d}\xi}+u\frac{\mathrm{d}u\gamma}{\mathrm{d}\xi}+\frac{\sigma}{n}\frac{\mathrm{d}p}{\mathrm{d}\xi}=-\frac{\mathrm{d}\phi}{\mathrm{d}\xi}, \tag{9}$$

$$\frac{\mathrm{d}p}{\mathrm{d}\xi}+u\frac{\mathrm{d}p}{\mathrm{d}\xi}+3p\frac{\mathrm{d}u}{\mathrm{d}\xi}=0, \tag{10}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}=ne-n. \tag{11}$$

Integrating equation (8) and using the boundary conditions $n\to 1$, $u\to u_1$, we get

$$n=\frac{v-u_1}{v-u}, \tag{12}$$

where u_1 is the drift velocity of the ion. Now from (11), eliminating ϕ, n, and p in terms of u, (also using the boundary conditions $\xi\to 0$, $n\to 1$, $p\to 1$) we get,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2}=\frac{\partial\psi}{\partial u}, \tag{13}$$

where

$$\psi = -\frac{g_e(u)+g_i(u)}{(v-u)^2\left[\gamma^3 - \frac{3\sigma(v-u_1)^2}{(v-u)^4}\right]^2}, \tag{14}$$

and $g_e(u)$ and $g_i(u)$ are given by

$$g_e(u) = (\mu+\nu\beta)\Big[\mu(e^{\frac{v_1}{\mu+\nu\beta}}-1) + \frac{\nu}{\mu+\nu\beta}(e^{\frac{\beta v_1}{\mu+\nu\beta}}-1)\Big], \tag{15}$$

$$g_i(u) = -\Big[vu\gamma - vu_1\gamma_1 + \sigma(v-u_1)^3\Big(\frac{1}{(v-u_1)^3} - \frac{1}{(v-u)^3}\Big)\Big], \tag{16}$$

where

$$v_1 = (vu-c^2)\gamma - (vu_1-c^2)\gamma_1 + \frac{3\sigma}{2}\Big[1-\frac{(v-u_1)^2}{(v-u)^2}\Big], \tag{17}$$

$$\gamma_1 = \frac{1}{\sqrt{1-u_1^2/c^2}}. \tag{18}$$

Thus

$$\frac{d^2u}{d\xi^2} = \frac{1}{(v-u)\left[\gamma^3 - \frac{3\sigma(v-u_1)^2}{(v-u)^4}\right]}\left(\mu e^{\frac{v_1}{\mu+\nu\beta}} + \nu e^{\frac{\beta v_1}{\mu+\nu\beta}} - \frac{v-u_1}{v-u}\right) + \frac{2}{(v-u)^3\left(\gamma^3 - \frac{3\sigma(v-u_1)^2}{(v-u)^4}\right)^2}(g_e(u)+g_i(u))$$
$$- \frac{2\left(\frac{3u\gamma^5}{c^2} - \frac{12\sigma(v-u_1)^2}{(v-u)^5}\right)}{(v-u)^2\left(\gamma^3 - \frac{3\sigma(v-u_1)^2}{(v-u)^4}\right)^3}(g_e(u)+g_i(u)). \tag{19}$$

Considering the terms of $O(\frac{1}{c^2})$ and $O(\sigma)$, we get

$$\psi = \frac{g(u)}{(v-u)^2}\left(1+\frac{3u^2}{c^2}+\frac{6\sigma(v-u_1)^2}{(v-u)^4}\right), \tag{20}$$

$$\frac{d^2u}{d\xi^2} = \frac{[\nu e^{C_1 v_1} + \mu e^{C_2 v_1}] - \frac{v-u_1}{v-u}}{(v-u)}\left(1-\frac{3u^2}{2c^2}+\frac{3\sigma(v-u_0)^2}{(v-u)^4}\right) + \left(\frac{2}{(v-u)^3} - \frac{6uv}{c^2(v-u)^3} + \frac{36\sigma(v-u)^2}{(v-u)^7}\right)g(u), \tag{21}$$

where

$$g(u) = uv\left(1+\frac{u^2}{2c^2}\right) - vu_1\left(1+\frac{u_1^2}{2c^2}\right) + \sigma\left(1-\frac{(v-u_1)^3}{(v-u)^3}\right) + (\mu+\nu\beta)\mu[e^{c_2 v_{11}}-1] + \frac{\nu}{\beta}[e^{c_1 v_{11}}-1], \tag{22}$$

$$v_{11} = vu\left(1+\frac{u^2}{2c^2}\right) - vu_1\left(1+\frac{u_1^2}{2c^2}\right) - \frac{u^2}{2} - \frac{3u^4}{8c^2} + \frac{u_1^2}{2} + \frac{3u_1^4}{8c^2}, \tag{23}$$

$$C_1 = \frac{\beta}{\mu+\nu\beta}, \tag{24}$$

$$C_2 = \frac{1}{\mu+\nu\beta}. \tag{25}$$

One can also write

$$\psi = \frac{(u')^2}{2}. \tag{26}$$

Considering single species of electrons ($\mu = 1$, $\nu = 0$, $\beta = 1$), and neglecting the relativistic effect ($u_1/c = 0$),

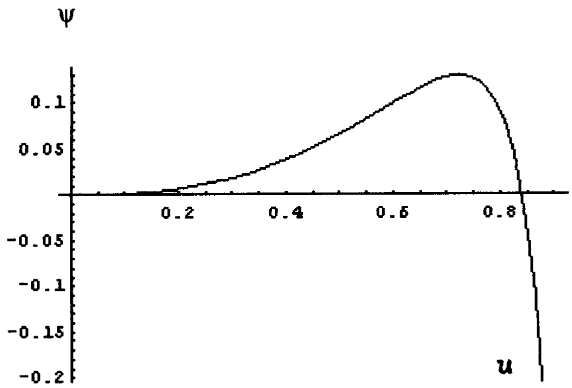

Fig. 1 ψ is plotted against u. The parameters are $v = 1.25$, $\mu = 0.5$, $\nu = 0.5$, $\beta = 0.05$, $\sigma = 0.001$, and $c = 100$.

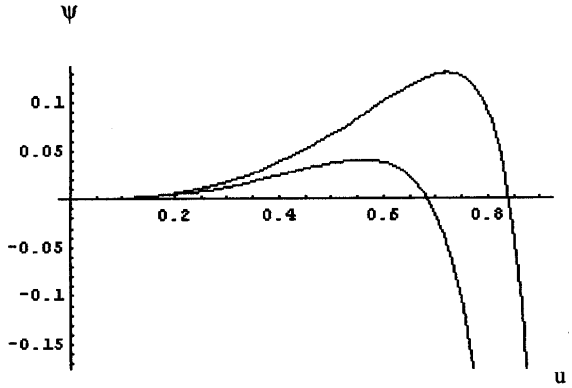

Fig. 3a. ψ is plotted against u for different values of v, viz. $v = 1.2$ and 1.25. Other parameters are same as those in Figure 1.

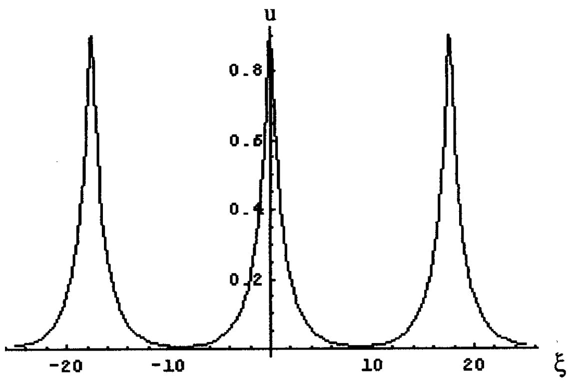

Fig. 2a. u is plotted against ξ for $u_0 = 0.841314$. Other parameters are same as those in Figure 1.

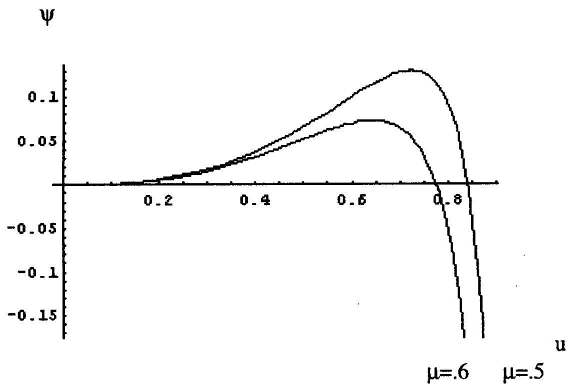

Fig. 3b. ψ is plotted against u for different values of μ, viz. $\mu = 0.5$ ($\nu = 0.5$), and 0.6. Other parameters are same as those in Figure 1.

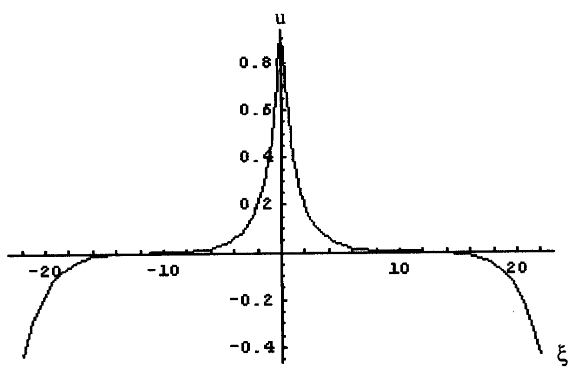

Fig. 2b. u is plotted against ξ for $u_0 = 0.841315$. Other parameters are same as those in Figure 1.

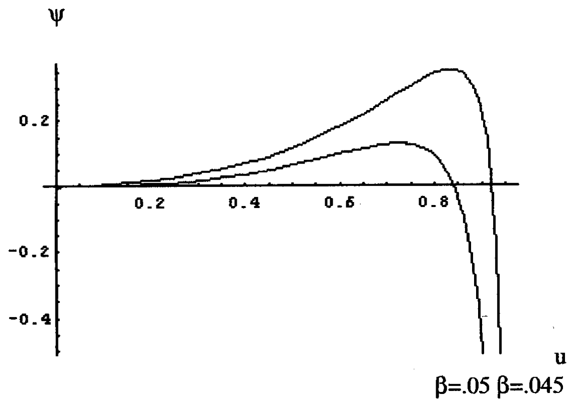

Fig. 3c. ψ is plotted against u for different values of β, (=0.045, 0.05). Other parameters are same as those in Figure 1.0

drift velocity of the ion ($u_1 = 0$) and ion temperature ($\sigma = 0$), Eq. (20) and (21) reduce to Eq. (22) and (23) of [26].

3. Results and Discussions

To find the region of existence of solitary waves one has to study the nature of the functions $\psi(u)$ and $\phi_1(u)$, defined by

$$\psi(u) = \frac{(u')^2}{2}, \tag{27}$$

where

$$u'' = \frac{\partial\psi}{\partial u} = \phi_1(u). \tag{28}$$

For solitary wave (see [29]) $\phi_1(u)$ will have two roots, one being at $u = 0$ and other at some point $u = u_2$ (≥ 0). Also $\phi_1(u)$ should be positive in the interval $(0, u_2)$ and negative in $(u_2, u_{\max})$, where $u_{\max} = u_0$ is obtained from the nonzero root of $\psi(u)$. To get the shape of the travelling solitary wave one has to solve $\psi_1(u) = u''$ numerically with suitable boundary conditions.

Figure 1 shows the plot of ψ vs. u with $v = 1.25$. Other parameters are $\sigma = 0.001$, $\mu = 0.5$, $\nu = 0.5$, $u_1 = 0.01$ and $\beta = 0.05$. It is seen that $\psi(u)$ crosses the u axis at $u = u_0 = 0.841314$. Hence the amplitude of the solitary wave for this set of parameters will be 0.841314. To get the shape of the solitary wave we have solved numerically $u'' = \psi_1(u)$ with $u_0 = 0.841314$, $u'_0 = 0$. Figure 2a depicts the soliton solution $u(\xi)$ plotted against ξ. Other parameters are same as those in Figure 1. It is seen that $u_0 = 0.841314$ is the critical value for u. For $u_0 > 0.841314$ the soliton solution ceases to exist and it is shown in Figure 2b. In this figure u_0 is taken as 0.841315. All the other parameters are same as those in Figure 2a. Here it is seen that a very small change (0.000001) in u_0 destroys the periodic behaviour of the solitary wave. Hence it is seen that such a small change in u_0 can destroy the periodic behaviour of the solitary wave. In [26] (where a single species of electron was considered and the ion temperature and relativistic correction of ion velocity were neglected) it was shown that for $v = 1.25$ the periodic behaviour of the soliton breaks at $u_0 = 0.711604$, but in this case the periodic behaviour breaks at $u_0 = 0.841314$, provided one considers the present model. To see the effect of the parameters v, μ or β on the critical values, Figs. 3a, b and c are drawn. In Fig. 3a ψ is plotted against u for $v = 1.2$ and 1.25. Other parameters are same as those in Figure 1. Here it is seen that the critical value of breaking of the soliton solution increases with the increase of v. In Fig. 3b ψ is plotted against u for $\mu = 0.5 (\nu = 0.5)$, $0.6 (\nu = 0.4)$. The other parameters are same as those in Figure 1. In this case the critical value of breaking of the soliton solution decreases with increase of μ, the cold electron concentration. In Fig. 3c ψ is plotted against u for $\beta = 0.05$, 0.045. The other parameters are same as those in Figure 1. In this case also the critical value of breaking of the soliton solution decreases with the increase of β, the ratio of cold and hot electron temperatures. Hence the soliton velocity v, electron concentration or ratio of cold and hot electron temperature all play significant roles in the forming and breaking of the solitary waves.

4. Conclusions

Using the pseudopotential approach, we have studied the speed and shape of the solitary waves. Sagdeev's potential is obtained in terms of u, the ion velocity. Considering a single species of electrons neglecting the ion temperature, the relativistic correction of the ion velocity, and the ion-drift velocity, our result reproduces the result obtained by Johnston and Epstein [26]. It is seen that there exists a critical value of u at which $u'^2 = 0$, beyond which the soliton solution would not exist. This critical value is extremely sensitive to other parameters and also depends on the soliton velocity. So it is seen that parameters like the soliton velocity, cold and hot electron concentration or ratio of cold and hot electron temperature all play significant roles in the forming and breaking of the solitary wave.

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